Midterm 2013 - Solutions

1 Question 1

Given the following regular expression:

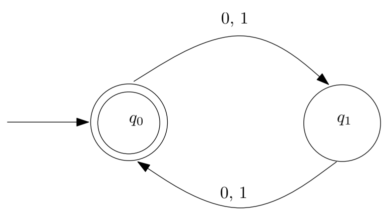
$$(00 + 01 + 10 + 11)^*$$

1. Define the language that corresponds to the expression.

$$L = \{w \mid |w| mod 2 = 0\}$$

2. Define a DFA that accepts the language

   (a) In a drawing



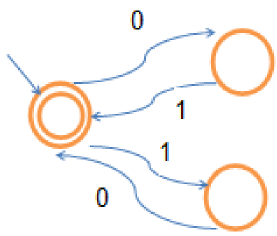
   (b) Formally

$$A = (Q, \Sigma, \delta, q_0, F)$$

Where, $Q = \{q_0, q_1\}$, $\Sigma = \{0, 1\}$, $q_0 = q_0$, $F = \{q_0\}$,
δ: $\delta(q_0, 0) = \delta(q_0, 1) = q_1$ and $\delta(q_1, 0) = \delta(q_1, 1) = q_0$

2 Question 2

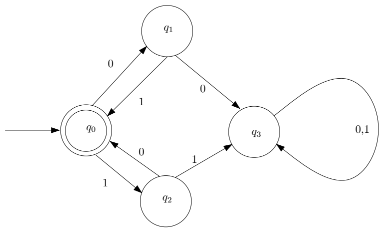
Given the following NFA,



1. Give a regular expression that corresponds to the language accepted by the NFA.

$$(01 \cup 10)^*$$

2. Draw a DFA with the minimal number of states that accepts the same language



3 Question 3

Show that the following language is not a CFL by the pumping lemma,

$$\{w \in 0, 1, 2^* \mid \#_0(w) = \min\{\#_1(w), \#_2(w)\} \text{ and } \#_0(w) < \max\{\#_1(w), \#_2(w)\}\}$$

For a pumping constant p, we will choose $w = 0^p 1^p 2^{p+1}$

For a subdivision of $w = uvxyz$ where $|vxy| \leq p$, $|vy| > 0$ and $vy \in 0^p$ (contains only 0's),
We will choose a pumping value of $i = 0$.
Clearly the number of 0's was reduced and is lower than p. Thus it is less than the number of 1's (which is p) and it is less than the number of 1's (which is $p + 1$) and hence is less than the minimum.

For a subdivision of $w = uvxyz$ where $|vxy| \leq p$, $|vy| > 0$ and $vy \in 0^p 1^p$ (contains 0's and 1's),
We will choose a pumping value of $i = 3$.
Clearly the number of 0's was increased by at least two and is larger than $p + 1$. Thus it is more than the number of 2's (which is $p + 1$) which is the minimum.

For a subdivision of $w = uvxyz$ where $|vxy| \leq p$, $|vy| > 0$ and $vy \in 1^p$ (contains only 1's),
We will choose a pumping value of $i = 0$.
Clearly the number of 1's was decreased and is smaller than p. Thus the number of 0's (which is equal to p) is more than the number of 1's which is the minimum.

For a subdivision of $w = uvxyz$ where $|vxy| \leq p$, $|vy| > 0$ and $vy \in 1^p 2^p$ or $vy \in 2^p$(contains at least one 2),
We will choose a pumping value of $i = 0$.
Clearly the number of 2's was decreased and is smaller or equal to p. The number of 1's is at most p. Thus the number of 0's (which is equal to p) is more or equal to the maximal number between 1's and 2's.

4 Question 4

We are given the following CFG: $S \to aSb|A|B$
$A \to aA|a$
$B \to bB|b$
What is $L(G)$?

$$L(G) = \{a^n b^m \mid n \neq m,\ n, m \geq 0\}$$

5 Question 5

We are given the following CFG:
$S \to 0S0|1S1|SS|\epsilon$
Is the following statement true? (prove your answer)

$$L(G) = \{ww^R \mid w \in \Sigma^*\}$$

The claim is not correct. We will show this by creating the word 0011 from the grammar where clearly $0011 \notin \{ww^R \mid w \in \Sigma^*\}$.

$S \to^a SS \to^b 0S0S \to^c 0S01S1 \to^d 0011$
a: using the rule $S \to SS$
b: using the rule $S \to 0S0$
c: using the rule $S \to 1S1$
d: using the rule $S \to \epsilon$ (twice)

6 Question 6

Since L_1 is regular there is a DFA $D =< Q, \Sigma, \delta, q_0, F >$ such that $L(D) = L_1$.
To prove that $L_1 \Box L_2$ is regular we define an NFA $N =< Q_N, \Sigma, \delta_N, q_N, F_N >$ such that $L(N) = L_1 \Box L_2$:

- One possible way:
  The starting state of N is the subset of Q reachable from q_0 by words in L_2:
  $Q_N = Q$, $\delta_N = \delta$, $q_N = \{q \in Q \mid \exists x \in L_2, \delta(q_0, x) = q\}$, $F_N = F$
- Second possible way:
  ϵ transitions from the new starting state q_N to the subset of Q reachable from q_0 by words in L_2:
  $Q_N = Q \cup \{q_N\}$, $\delta_N = \delta \cup \{((q_N, \epsilon), \{q \in Q \mid \exists x \in L_2, \delta(q_0, x) = q\})\}$, $F_N = F$
- Other ways are possible...

Correctness (for the first way above): if $y \in L_1 \Box L_2$ then there exists $x \in L_2$ such that $xy \in L_1$. Therefore, the path in N starting at the state of Q reachable by x reaches an accepting state in F when parsing y. For the other direction: if $y \in L(N)$ then there is a path in N that reaches an accepting state in F when parsing y and starting form a state $q \in q_N$. By the definition of q_N, there exists $x \in L_2$ such that $\delta(q_0, x) = q$ and we conclude that that $\delta(q_0, xy) \in F$ and therefore $xy \in L_1$ and $y \in L_1 \Box L_2$.