Computational Models, Spring 2013 Exrecise #6

$\mathcal{NP}$-Complete languages

By "graph", we mean undirected graph. By "number", we mean natural number.

1. For the following decision problems, determine whether they are in $\mathcal{P}$ or in $\mathcal{NPC}$ (assuming $\mathcal{P} \neq \mathcal{NP}$). Prove your answer.
   (a) Input: sets $A_1 \dots A_n$, and a number k.
   Question: does there exist a set C of size k, such that for every $1 \leq i \leq n$ $A_i \cap C \neq \emptyset$?
   (b) Input: a 3CNF formula ψ Question: does there exist an assignment that satisfies ψ and gives **True** for exactly 10 variables?
   (c) Input: a 3CNF formula ψ
   Question: do there exist at least two assignments that satisfy ψ?
   (d) Input: graph G.
   Question: does there exist a Hamiltonian path in G (between any pair of vertices)?
   (e) Input: graph G and a number k.
   Question: does there exist a simple path in G of length $\geq k$?
   (f) Input: graph G and a number k.
   Question: is there a Vertex-Cover S in G of size k and an IndependentSet, T, of size $\frac{k}{2}$, such that $T \subseteq S$?
2. Prove that if $\mathcal{P} = \mathcal{NP}$ then every language in $\mathcal{P}$, except $\emptyset$ and Σ^* is $\mathcal{NPC}$. Why can't $\emptyset$ and Σ^* be $\mathcal{NPC}$.
3. A c-coloring of an undirected graph $G = (V, E)$ is a map $\chi : V \to \{1 \dots c\}$ such that adjacent vertices (u, v) get different colors i.e. $\chi(u) \neq \chi(v)$. Let
$$c - COLORING = \{G \ : \ G \text{ is a graph that can be } c\text{-colored}\}$$
   (a) Prove that 3-coloring is $\mathcal{NPC}$.
   (b) Prove that 2-coloring is in $\mathcal{P}$.
4. Define $SETCOVER$ to be
$$\{(U, S_1, \dots S_m, k) \text{ s.t. } \forall i S_i \subset U \text{ and there is } I \subseteq \{1 \dots m\} \text{ with } |I| = k \text{ and } U = \bigcup_{i \in I} \{S_i\}\}$$
   Show that $SETCOVER$ is $\mathcal{NPC}$.