Computational Models - Lecture 6[1]

Handout Mode

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April 17/29, 2013

[1] Based on slides by Benny Chor, Tel Aviv University, modifying slides by Maurice Herlihy, Brown University.

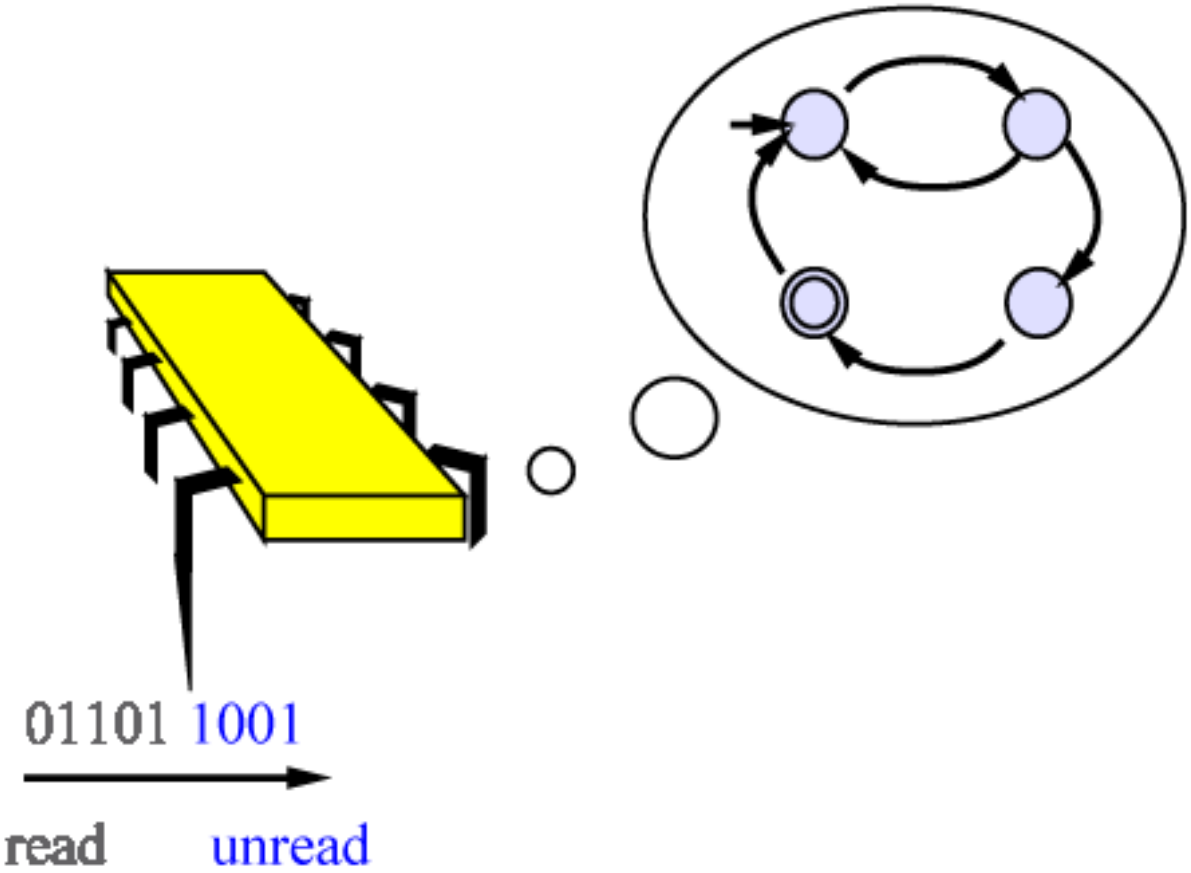
Talk Outline

- Turing Machines

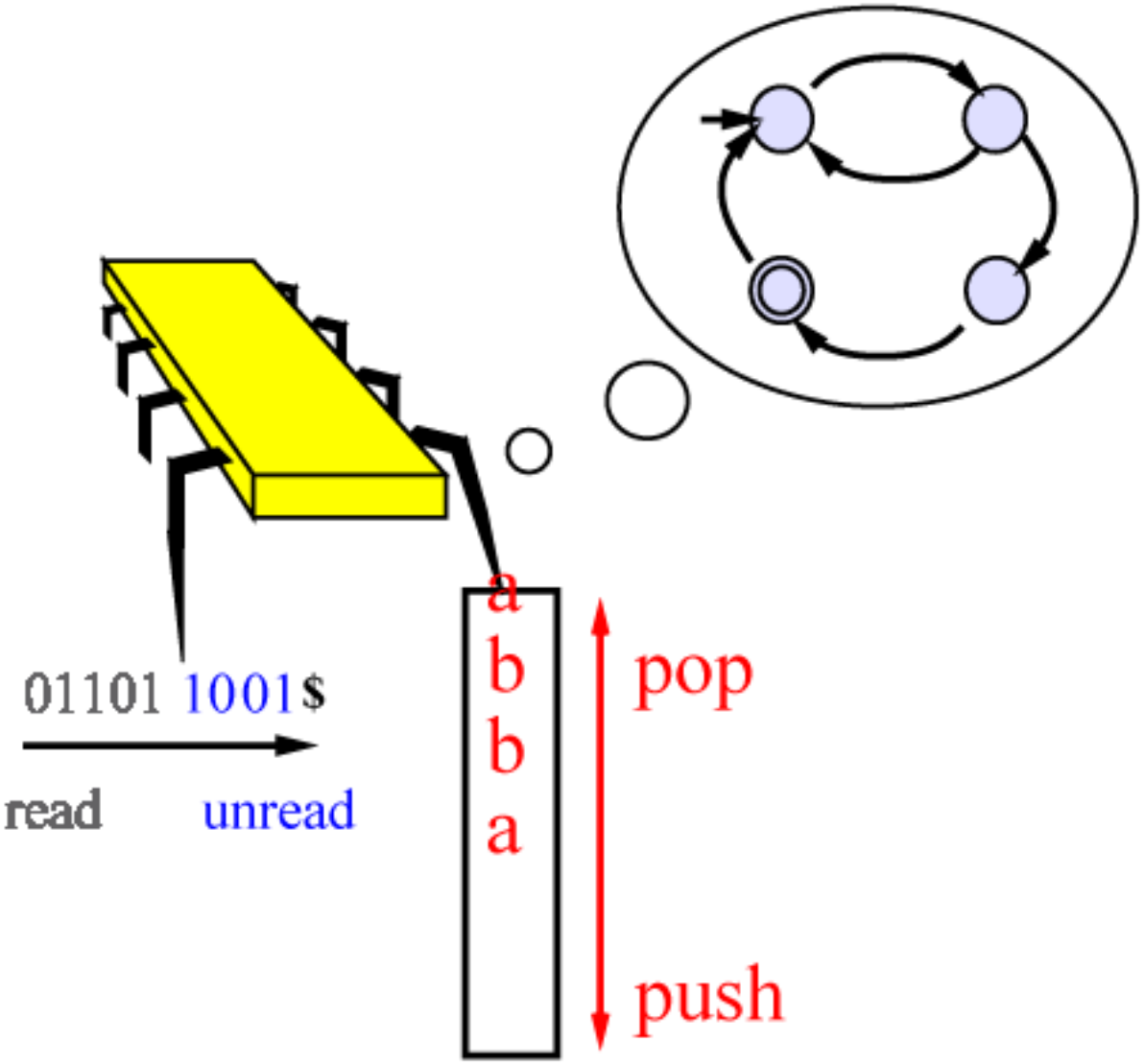
- Alternative Models of Computers
- Multitape TMs, RAMs, Non Deterministic TMs
- The language classes $\mathcal{R} = \mathcal{RE} \cap co\mathcal{RE}$

- Sipser's book, 3.1, 3.2, & 3.3

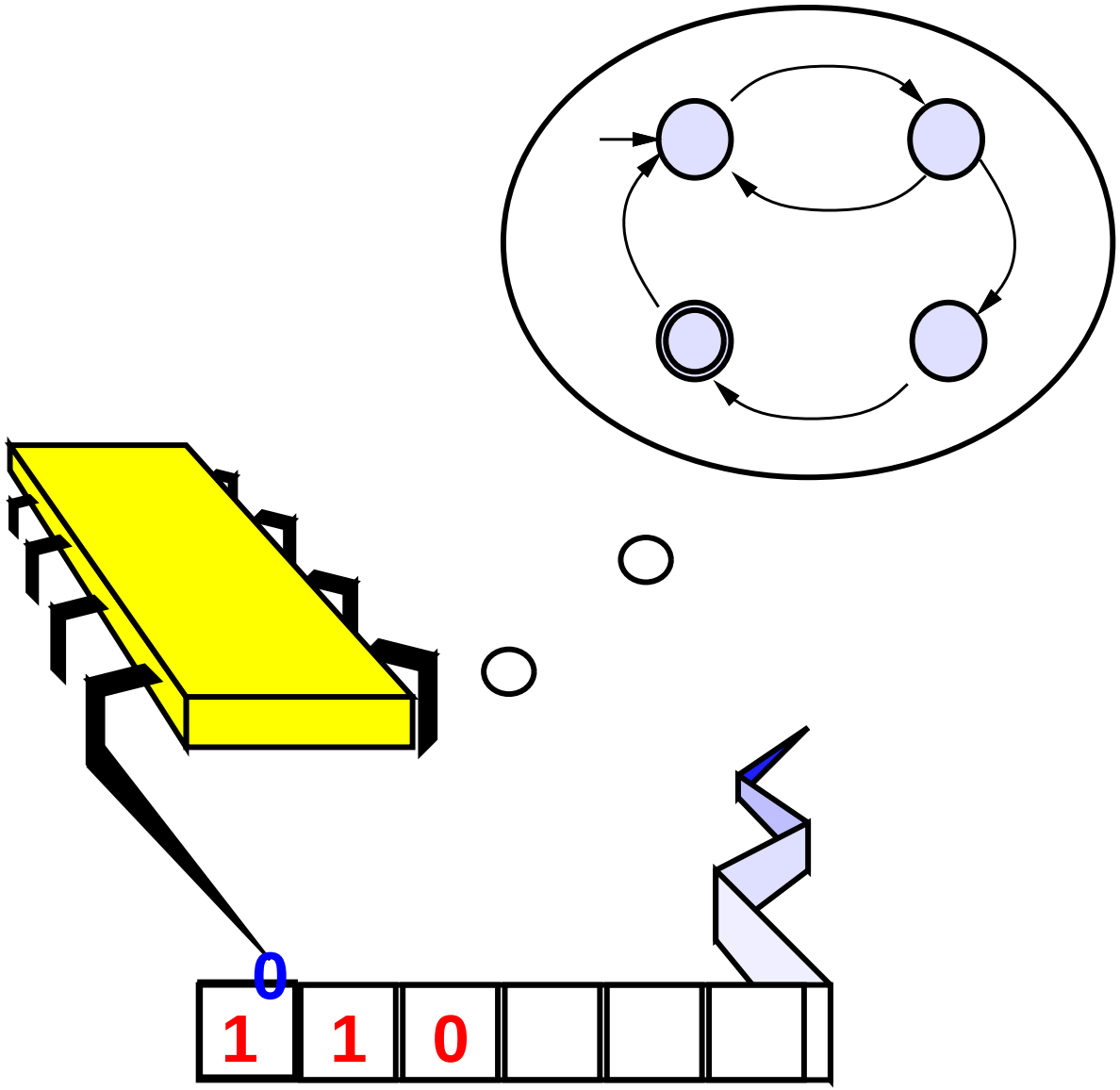
A Finite Automaton



A Pushdown Automaton



A Turing Machine



Part I

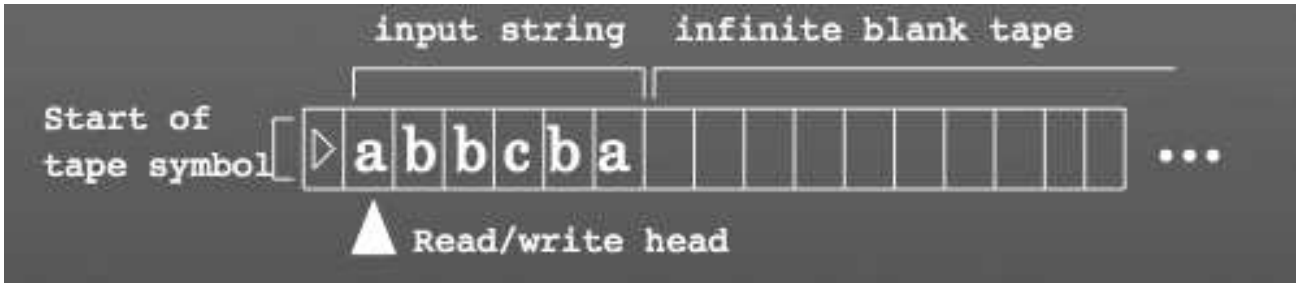
Turing Machines (TMs)

Turing Machines

- Machines so far (DFA, PDA) read input only once
- Turing Machines
  - Can back up over the input
  - Can overwrite the input
  - Can write information on tape and come back to it later

Turing Machines

- Input string is written on a tape:



- At each step, machine reads a symbol, and then
  - writes a new symbol, and
  - either moves read/write head to right,
  - or moves read/write head to left

TM vs. PDA vs. DFA

- A Turing machine can both write on the tape and read from it.
- A PDA is restricted to reading from the stack in LIFO manner.
- A DFA has no media for writing anything – it must all be in its finite state.
- The TM read-write head can move both to the left and to the right
- The TM read-write tape is infinite to the right
- The special final (accepting/rejecting) states of TM take immediate effect (so the head need not be at some special position)

Effects of A Single Step

- Changes current state,
- Changes head position and tape content at current position.

- Each step has very local, small effect.
- But many small effects can accumulate to a meaningful task.

Example $B = \{w\#w | w \in \{0,1\}^*\}$

Algorithm 1 (A TM deciding B (pseudocode))

1. Check for a single $\#$,
   - If false then reject.
2. Zig-zag across the tape, check identical letters, and replace them by X.
   - If not identical then reject.
3. When all the letters left of $\#$ are marked X, check for remaining letters right of $\#$
   - If there remaining letters are reject,
   - otherwise accept

Example $B = \{w\#w | w \in \{0,1\}^*\}$ cont.

- Input: $0\ 1\ 1\ 0\ 0\ 0\ \#\ 0\ 1\ 1\ 0\ 0\ 0$
- $\bar{0}\ 1\ 1\ 0\ 0\ 0\ \#\ 0\ 1\ 1\ 0\ 0\ 0$
- $X\ \bar{1}\ 1\ 0\ 0\ 0\ \#\ 0\ 1\ 1\ 0\ 0\ 0$
- $X\ 1\ 1\ 0\ 0\ 0\ \#\ \bar{0}\ 1\ 1\ 0\ 0\ 0$
- $X\ 1\ 1\ 0\ 0\ 0\ \bar{\#}\ X\ 1\ 1\ 0\ 0\ 0$
- $\bar{X}\ 1\ 1\ 0\ 0\ 0\ \#\ X\ 1\ 1\ 0\ 0\ 0$
- $X\ \bar{1}\ 1\ 0\ 0\ 0\ \#\ X\ 1\ 1\ 0\ 0\ 0$
- $X\ X\ \bar{1}\ 0\ 0\ 0\ \#\ X\ 1\ 1\ 0\ 0\ 0$
- $\ldots\ \ldots$
- $X\ X\ X\ X\ X\ X\ \#\ X\ X\ X\ X\ X\ X$

Turing Machine – Formal Definition

Definition 2 (Turing machine)

A Turing machine (TM) is a 7-tuple $(Q, \Sigma, \Gamma, \delta, q_0, q_a, q_r)$, where

- Q is a finite set of states,
- Σ the input alphabet not containing the blank symbol $\sqcup$
- Γ is the tape alphabet, where $\sqcup \in \Gamma$ and $\Sigma \subset \Gamma$.
- $\delta : Q \times \Gamma \to Q \times \Gamma \times \{L, R\}$ is the transition function,
- $q_0 \in Q$ is the start state,
- $q_a \in Q$ is the accept state, and
- $q_r \in Q$ is the reject state.

Transition Function δ

$$\delta : Q \times \Gamma \to Q \times \Gamma \times \{L, R\} .$$

where $\delta(q, a) = (r, b, L)$ means: in state q where head reads tape symbol a, the machine:

- writes b over a ($a = b$ is possible),
- enters state r,
- moves the head left (this is what the L stands for).

- If $\delta(q, a)$ is undefined, it means assume $\delta(q, a) = (q_r, \cdot, \cdot)$

Model of Computation

Before we start:

- an input of length n, $w = w_1 w_2 \ldots w_n \in \Sigma^*$ is placed on n leftmost tape squares, one tape square per input letter.
  Rest of tape contains blanks $\sqcup$
- since $\sqcup \notin \Sigma$, leftmost blank indicates end of input.
- read/write head is on leftmost square of tape

The "computation":
M "computes" according to transition function δ.
Computation continues until q_a or q_r is reached. Otherwise M runs forever.

- If M tries to move head beyond left-hand-end of tape, it doesn't move (still M does not crash)

TM Configurations

A TM configuration is a convenient notation for recording the state, head location, and tape contents of a TM in a given instant. Think of it as a snapshot.

- For example, configuration $1011q_70111$ means:
- Current state is q_7,
- left hand side of tape (to the left of the head) is 1011,
- right hand side of tape is 0111,
- and head is on 0 (leftmost entry of right hand side).

Configurations: The Yield Relation

- If $\delta(q_i, b) = (q_j, c, L)$, then configuration uaq_ibv yields configuration uq_jacv.
- If $\delta(q_i, b) = (q_j, c, R)$, then configuration uaq_ibv yields configuration $uacq_jv$.
- Special case (1): when head is at left end and tries to move left, it changes state and writes on tape but does not move, so if $\delta(q_i, b) = (q_j, c, L)$, configuration q_ibv yields q_jcv.
- Special case (2): What happens when head is at right end? We let wq_i and $wq_i\sqcup$ denote the same configuration, so moves to the right can now be accomodated.

  The new configuration is longer as it "annexed" one blank. This allows configurations to grow in length with computation. Yet at any given moment, they are finitely long.

Special Configurations

- Starting configuration: $q_0 w$
- Accepting configuration: $w_0 q_a w_1$
- Rejecting configuration: $w_0 q_r w_1$
- Halting configurations: $w_0 q_a w_1$ and $w_0 q_r w_1$

Accepting a Language

A Turing machine M accepts an input w, if there is a sequence of configurations $C_1, C_2, \ldots, C_k$ such that

- C_1 is the start configuration of M on w,
- each C_i yields C_{i+1},
- C_k is an accepting configuration.

The collection of strings accepted by M is called the language of M, and is denoted $L(M)$.

Enumerable Languages

Definition 3

A language L is called recursively enumerable ($\mathcal{RE}$), if some Turing machine accepts L.
(In the book called Turing-recognizable)

On an input, w, a TM may

- accept
- reject
- loop (run forever)

Major concern: In general, we never know if the TM will halt.

Decidable Languages

Definition 4

A TM M decides a language L, if it accepts it, and M halts on every input in Σ^*.

Namely the TM either reaches state q_a (in case $w \in L(M)$) or it reaches state q_r (in case $w \notin L(M)$), but it does not loop.

Such TM is called a decider.

Definition 5

A language L is decidable ($\mathcal{R}$), if some Turing machine decides it.
(In the book called Turing-decidable, also known as, recursive)

Knowing when Reaching the Left End

Question 6

To implement algorithms, we sometimes should be able to tell when a TM is at the left end of the tape.

Answer: Mark it with a special symbol. When head reads this symbol, TM “knows” it is on the leftmost tape square.

Part II

Some Examples

Example $A = \{0^{2^n} : n \geq 0\}$

Algorithm 7 (A TM deciding A (pseudocode))

On input w:

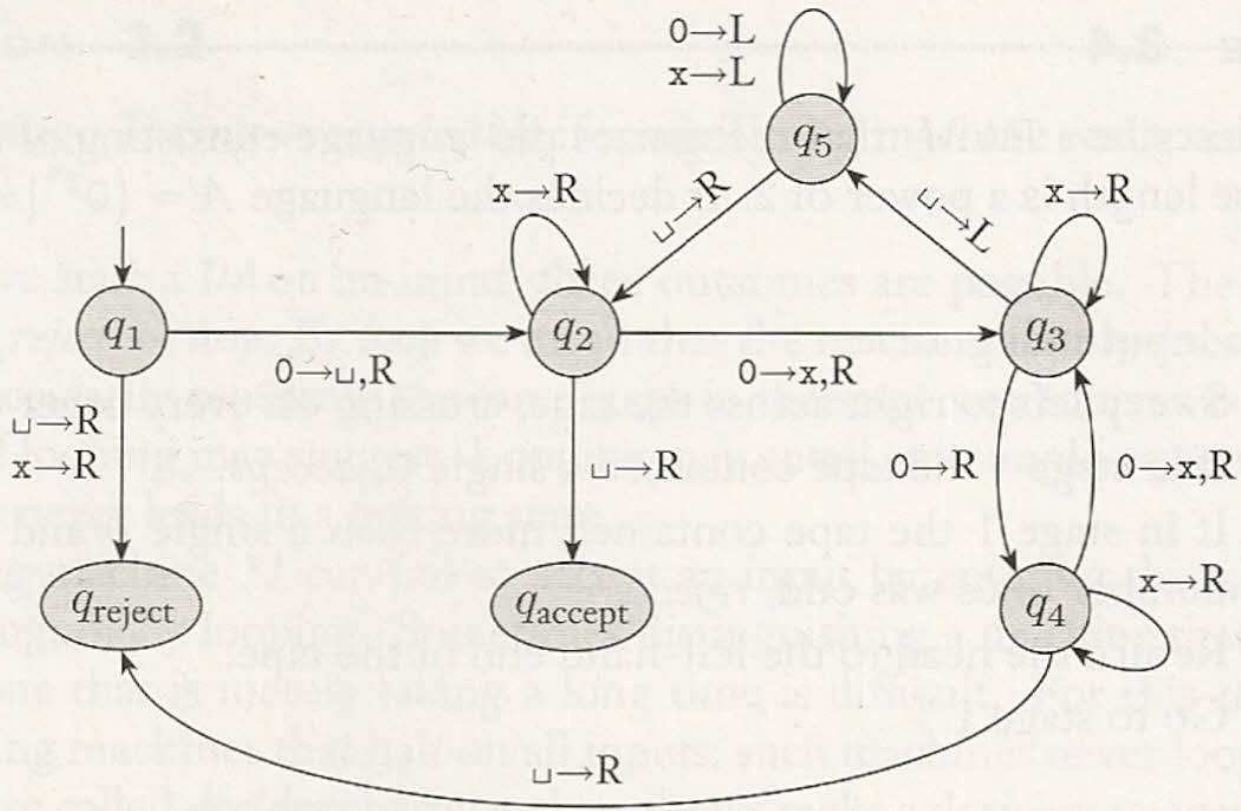
1. Move to the right, erasing every other 0
2. Accept if tape contains a single 0.
3. Reject if tape contains an odd number of 0 (greater than one).
4. Return to the start of the tape, and go to Step 1

Example $A = \{0^{2^n} : n \geq 0\}$ – TM formal definition

Definition 8 ($M_2 = (Q, \Sigma, \Gamma, \delta, q_1, q_a, q_r)$)

- $Q = \{q_1, q_2, q_3, q_4, q_5, q_a, q_r\}$
- $\Sigma = \{0\}$ input alphabet
- $\Gamma = \{0, x, \sqcup\}$ tape alphabet
- q_1 start state
- q_a accept state
- q_r reject state

Example $A = \{0^{2^n} : n \geq 0\}$ – Transition Function

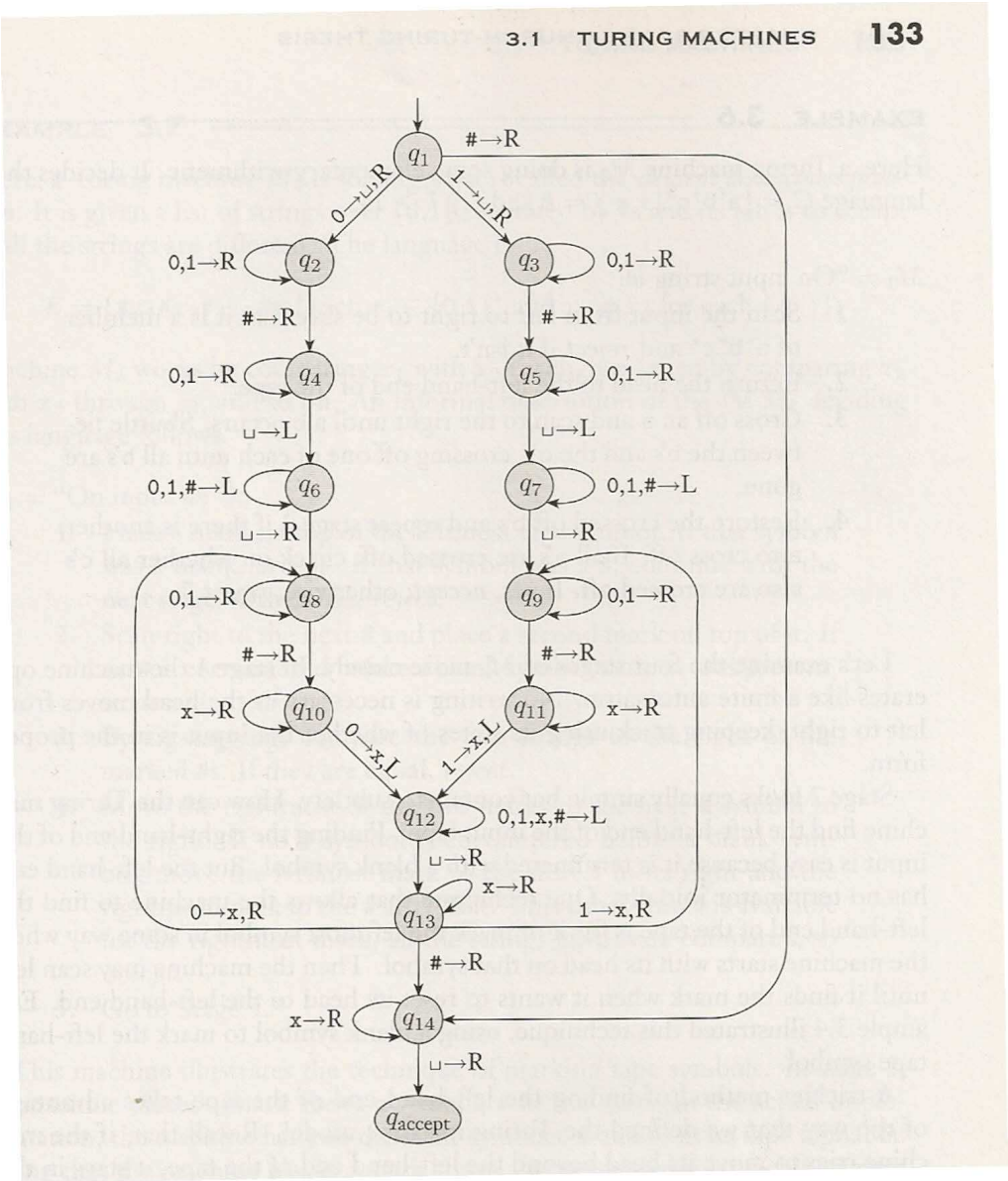


Example $B = \{w\#w \mid w \in \{0,1\}^*\}$ – TM formal definition

Definition 9 ($M_1 = (Q, \Sigma, \Gamma, \delta, q_1, q_a, q_r)$)

- $Q = \{q_1, \ldots, q_{14}, q_a, q_r\}$.
- $\Sigma = \{0, 1, \#\}$ input alphabet
- $\Gamma = \{0, 1, \#, x, \sqcup\}$ tape alphabet
- q_1 start state
- q_a accept state
- q_r reject state

Example $B = \{w\#w : w \in \{0,1\}^*\}$ – Transition Function



Example: $C = \{a^i b^j c^k : i \cdot j = k \ \wedge \ i, j, k \geq 1\}$

Algorithm 10 (A TM deciding C (pseudocode))

1. Scan from left to right to check that input is $a^+b^+c^+$
2. Return to start of tape
3. Cross off one a and scan right until b occurs.
   Shuttle between b's and c's, crossing off one of each, until all b's are gone.
4. Restore the crossed-off b's and repeat previous step if more a's exist.
   If all a's crossed off, check if all c's crossed off. If yes, accept, otherwise reject.

Example: Element Distinctness

Consider the element distinctness problem

$$E = \{\#x_1\#x_2\#\ldots\#x_\ell : x_i \in \{0,1\}^* \wedge i \neq j \implies x_i \neq x_j\}$$

Verbally,

- List of strings in $\{0,1\}^*$ separated by $\#$'s.
- List is in language (& machine should accept), if all strings are different.

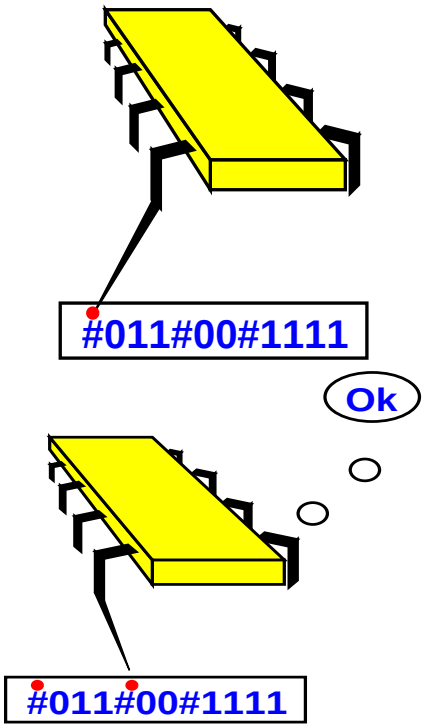
Decider for $E = \{\#x_1\#x_2\#\ldots\#x_\ell : x_i \in \{0,1\}^* \wedge i \neq j \implies x_i \neq x_j\}$

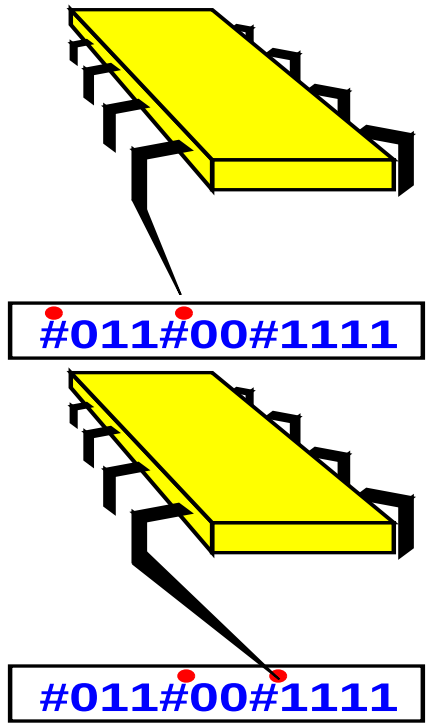
Algorithm 11 (A TM deciding E (pseudocode))

Input: w

1. Place a “mark" on leftmost tape symbol. If symbol not $\#$, reject.
2. Scan right to next $\#$ and place mark on top. If none, reject.
3. Compare the two strings to right of marked $\#$’s (how?). If equal, reject.
4. If possible, move rightmost mark to next $\#$ on right and go to Step 3.
5. If possible, move leftmost mark to next $\#$ on right and rightmost mark to $\#$ after that, and go to Step 3.
6. Accept.

Decider for $E = \{\#x_1\#x_2\#\ldots\#x_\ell : x_i \in \{0,1\}^* \wedge i \neq j \implies x_i \neq x_j\}$





Question 12

How do we “mark” a symbol?

Answer: For each tape symbol $\#$, add tape symbol $\dot{\#}$ to the tape alphabet Γ.

Part III

Alternative Definitions

Turing Machine Variants

For example, suppose the Turing machine head is allowed to stay put.

$$\delta : Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R, S\}$$

Question 13

Does this add any power?

Answer: No. Replace each S transition with two transitions: R then L. (ahmm ... why not vice-versa?)
Important notion here: Two-way emulation (model A capable of emulating model B; model B capable of emulating model A).

Multitape Turing Machines

- Constant number of infinite tapes
- Each tape has its own head
- Initially, input string on tape 1 and rest blank

For the transition function: $\delta : Q \times \Gamma^k \to Q \times \Gamma^k \times \{L, R\}^k$, the expression $\delta(q_i, a_1, \ldots, a_k) = (q_j, b_1, \ldots, b_k, L, R, \ldots, L)$ means:

Assuming the machine is in state q_i and heads 1 through k reading $a_1, \ldots, a_k$:

- the machine goes to state q_j,
- heads 1 thought k write $b_1, \ldots, b_k$,
- each head moves right or left as specified.

Equivalence Of Multitape TM and Singletape TM

Theorem 14

A singletape TM can emulate a multitape TM.

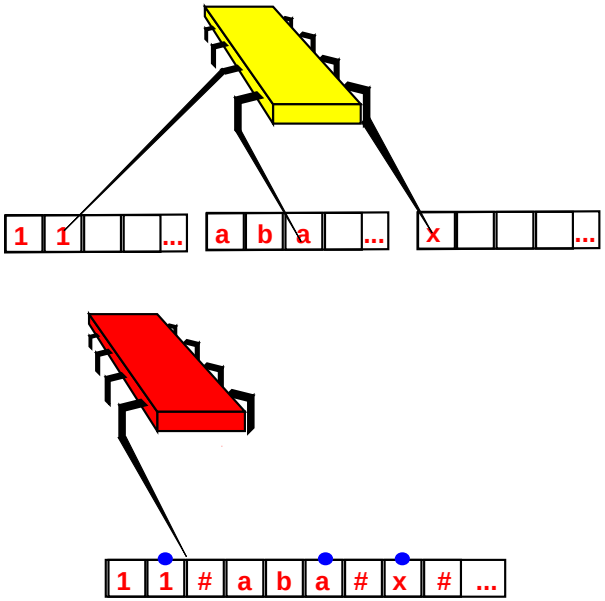
Corollary 15

A language is enumerable [resp., decidable], if and only if there is some multitape Turing machine that accepts [resp., decides] it.

One direction is trivial.

To prove the other direction, we will show how to convert a multitape TM, M, into an equivalent singletape TM, S.

Emulation Idea



- S emulates k tapes of M by storing them all on a single tape with delimiter $\#$.
- S marks the current positions of the k heads by placing $\bullet$ "above" the letters in current positions. It "knows" which tape the mark belongs to by counting (up to k) from the $\#$'s to the left.

The Emulator

Algorithm 16 (The Emulator (pseudocode))

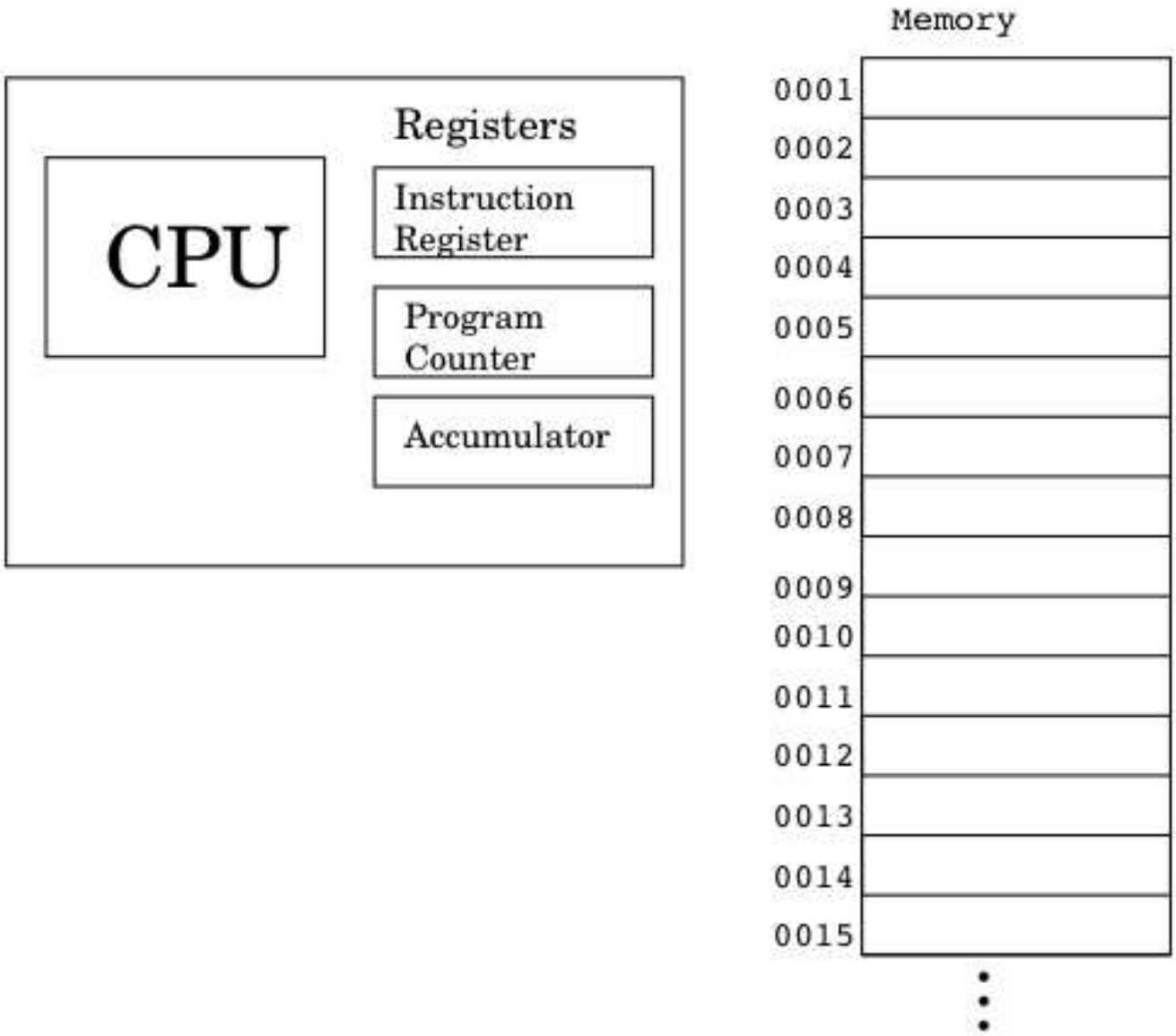
On input $w = w_1 \cdots w_n$:

1. Write on the tape $\# \overset{\bullet}{w}_1 \, w_2 \cdot w_n \# \overset{\bullet}{\sqcup} \# \overset{\bullet}{\sqcup} \# \cdots \#$
2. Scan the tape from first $\#$ to $(k+1)$-st $\#$ to read symbols under "virtual" heads.
3. Rescan the tape to write new symbols and move heads
4. When M moves head onto unused blank square, S will try to move virtual head onto $\#$.
   S handles it by writing blank $\sqcup$ on tape, and shifting the rest of tape one square to the right.

Random Access Machine (RAM)

- CPU
- 3 Registers (Instruction Register (IR), Program Counter (PC), Accumulator (ACC)
- Memory
- Operation:
  - Increment PC
  - Set IR $\leftarrow$ MEM[PC]
  - Execute instruction in IR
  - Repeat
- Instructions are typically compare, add/subtract, multiply/divide, shift left/right.
- We assume no limit on the registers' size
- All instructions are doable on a TM.

RAM: Schematic Picture



RAM: Instructions

	Instruction	Meaning
00	HALT	Stop Computation
01	LOAD x	ACC $\leftarrow$ MEM[x]
02	LOADI x	ACC $\leftarrow$ x
03	STORE x	MEM[x] $\leftarrow$ ACC
04	ADD x	ACC $\leftarrow$ ACC + MEM[x]
05	ADDI x	ACC $\leftarrow$ ACC + x
06	SUB x	ACC $\leftarrow$ ACC - MEM[x]
07	SUBI x	ACC $\leftarrow$ ACC - x
08	JUMP x	PC $\leftarrow$ x
09	JZERO x	PC $\leftarrow$ x if ACC = 0
10	JGT x	PC $\leftarrow$ x if ACC $>$ 0

RAM: Example Program

A program that multiplies two numbers (in locations 1000 & 1001), and stores the result in 1002

Memory	Machine Code	Assembly
0001	011000	LOAD 1000
0002	031003	STORE 1003
0003	020000	LOADI 0
0004	031002	STORE 1002
0005	021003	LOAD 1003
0006	090013	JZERO 0013
0007	070001	SUBI 1
0008	031003	STORE 1003
0009	011002	LOAD 1002
0010	041001	ADD 1001
0011	080004	JUMP 4
0013	000000	HALT

TM is (at least) as strong as RAM

Theorem 17

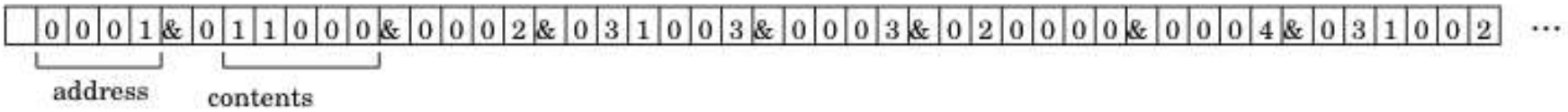
A Turing machine can emulate this RAM model.

Proof's idea: We can emulate the RAM model with a multi-tape Turing machine:

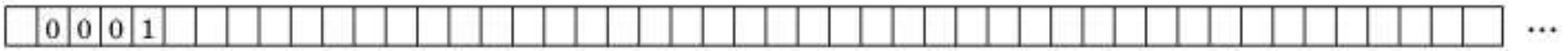
- One tape for each register (IR, PC, ACC)
- One tape for the Memory
- Memory tape will be entries of the form <address> <contents> in increasing order of addresses.

Emulator's Tapes

Memory

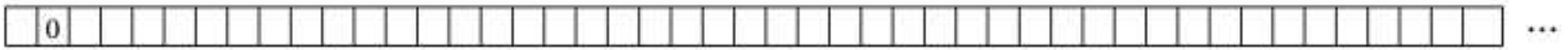


Instruction Pointer



Instruction Register

Accumulator



The Emulator

Algorithm 18 (Emulator (high-level description))

1. Scan through memory until reach an address that matches the PC
2. Copy contents of memory at that address to the IR
3. Increment PC
4. Based on the instruction code:
   - Copy a value into PC
   - Copy a value into Memory
   - Copy a value into the ACC
   - Perform an arithmetic operation, a shift, or a comparison

Beyond RAM

- A RAM can be modeled (emulated) by a Turing Machine.
- Any current machine (architecture, manufacturer, operating system, power supply, etc.) can be modeled by a Turing Machine.
- If there is an algorithm for it, a Turning Machine can do it.
- Note that at this point, we don't care how long it might take, just that it can be done.

Turing Completeness

- A computation model is called “Turing Complete” if it can emulate a (general) Turing Machine.
- Turing Complete $\Rightarrow$ can compute anything a TM could
  - Of course it might not be convenient ...
- Church-Turing Thesis: any computational model can be emulated by a Turing machine.

Part IV

Church-Turing Thesis

Computation Model

- Many models have been proposed for general-purpose computation. Remarkably, all “reasonable” models were shown to be equivalent to Turing machines.
- The notion of an algorithm is model-independent!
- We don’t really care about Turing machines *per se*.
- We do care about understanding computation, and because of their simplicity, Turing machines are a good model to use.

Models Equivalent to TM

- All “reasonable” programming languages (e.g., Java, Pascal, C, Python, Scheme, Mathematica, Maple, Cobol,. . .).
- λ-calculus of Alonzo Church
- Turing machines of Alan Turing
- Recursive functions of Godel and Kleene
- Counter machines of Minsky
- Normal algorithms of Markov
- Unrestricted grammars
- Two stack automata
- Random access machines (RAMs) ⋮

Church-Turing Thesis

"The intuitive notion of reasonable models of computation equals Turing machine algorithms".

Part V

Non-Deterministic Turing Machines

Non-Deterministic Turing Machines

Transition function:

$$\delta : Q \times \Gamma \rightarrow \mathcal{P}(Q \times \Gamma \times \{L, R\})$$

- Computation of a deterministic TM can be viewed as a path in configuration space.
- Computation of a non-deterministic TM (NTM) can be viewed as a tree in configuration space.
- NTM accepts an input if there is ($\exists$) an accepting branch in its computation tree.
- NTM rejects an input if all ($\forall$) branches in its computation tree are either rejecting or infinite (looping).

Equivalence of TM and NTM

Theorem 19

A Turing Machine can emulate a non-deterministic Turing machine.

Corollary 20

A language is enumerable [resp., recursive], if and only if there is some non-deterministic Turing machine that accepts [resp., decides] it.

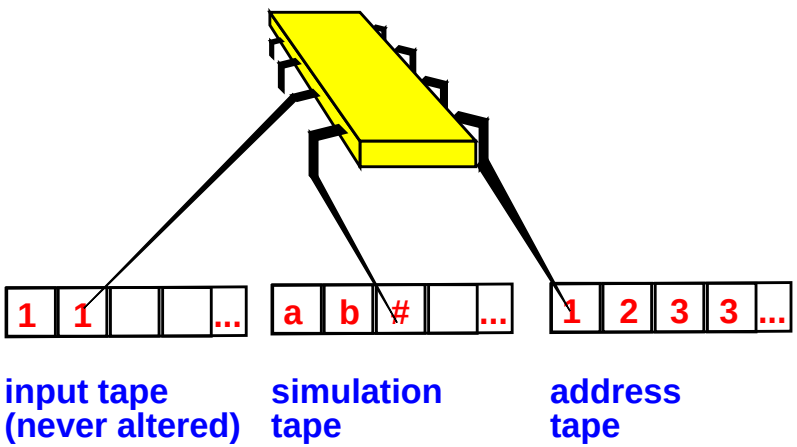
One direction is trivial.

To prove the other direction, we will show how to convert a non-deterministic TM, N, into an equivalent deterministic TM, D.

Basic idea

- D tries all possible branches
- If D finds any accepting branch, it accepts.
- If all branches reject, D rejects.
- If all branches reject or loop, D loops.

Emulator's Tapes



The emulating machine D has three tapes

- Input tape is never altered (only read from),
- Emulation tape serves as N's tape,
- Address tape keeps track of D's location in N's *computation tree*.

Computation Tree

We view N's non-deterministic computation as a tree in the configurations' space.

- Each tree branch is a branch of N's non-deterministic computation
- Each tree node is a configuration of N
- Root is starting configuration
- The number of children of each node, denoted by b, is at most the number of N's states, times the size of Γ, times 2 (left/right).

D traverses this tree until if finds an accepting configuration (if any).

Question 21

How to traverse this tree?

- depth-first search?
- breadth-first search?

Address Tape

The address tape contains a pointer into the configuration tree. Concretely, a string in Σ_b^*, where $\Sigma_b = \{1, \ldots, b\}$.

Definition 22

By incrementing the value of the address tape, we mean increasing its value by 1 (in base b).

Question 23

Can a TM implement the increment function?

Question 24

Can a TM compute the value of the node indexed by the address tape (with respect to to a given input w)?

The Emulator

Algorithm 25 (Emulator (pseudocode))

1. Compute the configuration of N indexed by the *address* tape :
   1. Copy *input* tape (i.e., w) to *emulation* tape.
   2. Use *emulation* tape to emulate the run of N on w, using the *address* tape to resolve non-deterministic choices.
      Break current emulation, if
      - ★ symbols on *address* tape are exhausted (i.e., end of branch)
      - ★ non-deterministic choice is invalid
      - ★ rejecting configuration was reached
2. Accept if accepting configuration reached.
3. Increment the value of *address* tape.
4. Go back to Step 1.