Computational Models - Lecture 13[1]

Handout Mode

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[1] Based on frames by Benny Chor, Tel Aviv University, modifying frames by Maurice Herlihy, Brown University.

Talk Outline

- Time Hierarchy Theorem (*Sipser*, 9.1)
- NP Hardness (*Sipser*, 10.1, 10.2)
  - Decision, search, and optimization problems
  - Approximation
  - Randomization
  - Fixed parameter algorithms
  - Heuristics
- Concluding Remarks

Part I

Time Hierarchy Theorem

Time Hierarchy Theorem

Definition 1 (time-constructible function)

Function $t : \mathbb{N} \mapsto \mathbb{N}$ is time constructible, if it is computable in time $O(t(n))$.

Theorem 2 (time hierarchy theorem)

For any time constructible function t with $t(n) \geq n \log n$, there exists a language L that is decidable in time $O(t(n))$ but not in time $o(t(n)/\log t(n))$.

Corollary 3

For any $1 \leq \varepsilon_1 < \varepsilon_2$ it holds that $\mathrm{DTIME}(n^{\varepsilon_1}) \subsetneq \mathrm{DTIME}(n^{\varepsilon_2})$.

Corollary 4

$\mathcal{P} \subsetneq \mathrm{DTIME}(n^{\log n})$ *and* $\mathcal{NP} \subsetneq \mathrm{DTIME}(2^{n^{\log n}})$.

Proof: $\mathcal{P} := \bigcup_{c \geq 0} \mathrm{DTIME}(n^c) \subseteq \mathrm{DTIME}(n^{\frac{1}{2} \cdot \log n}) \subsetneq \mathrm{DTIME}(n^{\log n})$

Proving the Time Hierarchy Theorem

Fix a time-constructible function t with $t(n) \geq n \log n$, and consider the following language L:

Algorithm 5 (D)

On input w:

1. Let $n = |w|$.
2. Store $Max = \lceil t(n)/\log t(n) \rceil$ in a binary counter.
3. `Reject` if w is not of the form $\langle M \rangle 10^*$ or if $|\langle M \rangle| > \log t(n)$.
4. Emulate M on w:
   - `Reject` if M makes more then Max steps (use the counter).
   - `Reject` if M accepts; `Accept` if M rejects.

Let $L = L(D)$ — the language decided by D.

Analyzing D's Running Time

Algorithm 6 (D)

On input w:

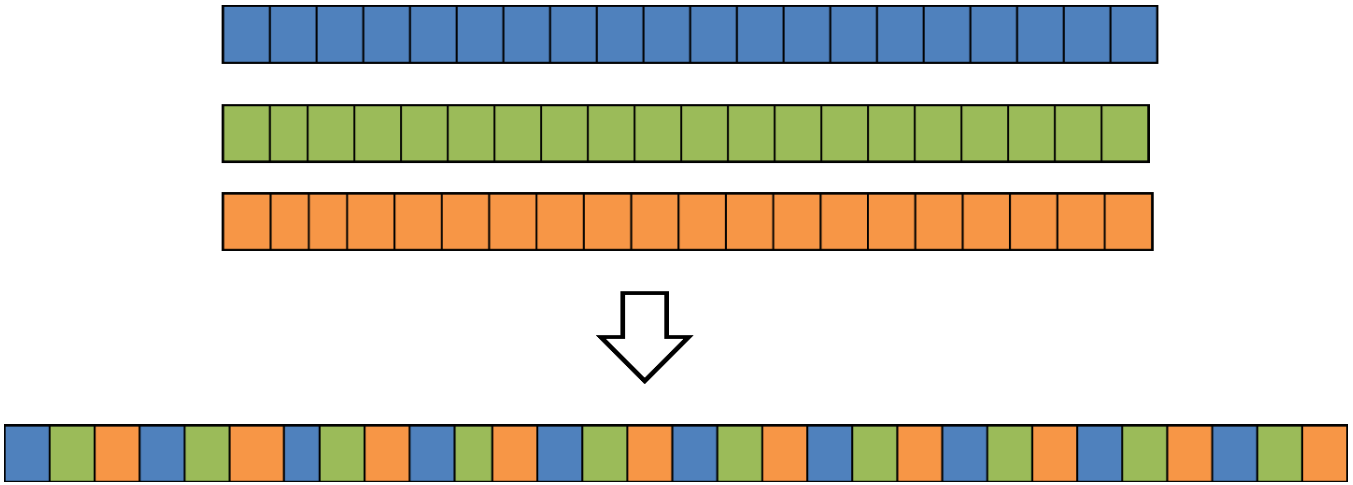
1. Let $n = |w|$.
2. Store the value $Max = \lceil t(n)/\log t(n) \rceil$ in a binary counter.
3. `Reject` if w is not of the form $\langle M \rangle 10^*$ or if $|\langle M \rangle| > \log t(n)$.
4. Emulate M on w:
   - `Reject` if M makes more then Max steps (use the counter).
   - If M accepts, `Reject`; If M rejects, `Accept`.

Claim 7

D runs in time $O(t(n))$.

- Clearly, steps $1, 2, 3$ can be performed in time $O(t(n))$.
- Step 4 could be performed in time $O(t(n))$ on a three-tape TM.
- Emulating $M(w)$ in time $O(t(n))$ on a single-tape TM (while keeping track on the counter), is more challenging...

Emulating M(w) in Time $O(t(n))$



1. We use 3 tracks stored interchangeably on the (single) tape.
   1. M's emulated tape
   2. Description of M's transition function
   3. The binary counter
2. We maintain the content of tracks $2,3$ near the head of the first track
3. Since the content of tracks $2,3$ is of length at most $\log t(n)$ (why?), the whole emulation + counter updates can be done in time $O(\log t(n) \cdot t(n)/\log t(n)) = O(t(n))$.

L is Not Decided in time $o(t(n)/\log t(n))$

Proof:

- Let M be a TM of running time $g(n) \in o(t(n)/\log t(n))$.
- $\exists\, n_0 \in \mathbb{N}$ such that $g(|w|) \leq t(|w|)/\log t(|w|)$ and $|\langle M \rangle| \leq \log t(n)$, where $w = \langle M \rangle 10^{n_0}$.
- $D(w)$ emulates $M(w)$ to its completion.
- Hence $D(w) \neq M(w)$, $\implies$ M is not a decider for L.

♣

Part II

NP Hardness

NP-Hard Languages

Definition 8 (NP-hard languages)

A language B is NP-hard, if it satisfies

- Every A $\in \mathcal{NP}$ is poly-time reducible to B

We do not require B $\in \mathcal{NP}$ (membership).

Example 9

The language L $= \{\langle \Phi_1, \Phi_2 \rangle : \Phi_1 \in \text{SAT}, \Phi_2 \notin \text{SAT}\}$ is NP-hard, but apparently not NP-complete. (why?)

Definition 10 (coNP-hard languages)

A language B is coNP-hard, if it satisfies

- Every A $\in$ *co*-$\mathcal{NP}$ is poly-time reducible to B

We do not require B $\in$ *co*-$\mathcal{NP}$ (membership).
The language L from above is coNP-hard but apparently not coNP-complete (why?).

NP-Hard Problems

Definition 11 (problems)

A problem is a set $T \subseteq \Sigma^* \times \Sigma^*$.
A (deterministic) algorithm S is a solver for T, if for all $x \in \Sigma^*$,
$S(x) = y \implies (x, y) \in T$
$S(x) = \perp \implies \nexists y \in \Sigma^*$ such that $(x, y) \in T$.

Deciding a language L is equivalent of solving the decision problem $L \times \{1\}$.

NP-Hard Problems, 2

Definition 12 (Karp reductions)

A language L is Karp reducible to a problem T, denoted $L \leq_P^{Karp} T$, if there exists a polynomial-time oracle-aided algorithm D, such that for any solver S for T, it holds that D^S is a decider for L.[a]

[a] We do not count the oracle running time, as part of D's running time.

Definition 13 (NP-hard problems)

A problem T is NP-hard, if for every $L \in \mathcal{NP}$ is holds that $L \leq_P^{Karp} T$.

Note that any NP-hard problem, is also coNP-hard.

Example 14 (#3SAT)

The #3SAT problem is: given a 3CNF formula ϕ as input, compute the number of satisfying assignment for ϕ.

- #3SAT is NP-hard
- It is likely "much harder" than NP.

Section 1

Decision, Search, and Optimization Problems

Decision Vs. Search Problems

Let $L \in \mathcal{NP}$.

- **Decision Problem:** Solving the problem $T = L \times \{1\}$
  — given and input x, decide if $x \in L$.
- **Search Problem:** Solving the problem
  $T = R_L := \{(x, y) : x \in L \wedge y \text{ is a certificate for } x\}$
  — given input x, find some y satisfying $R_L(x, y)$, or declare that none exist.

  The search problem seems harder to solve than the decision problem.
- Turns out that for NP complete languages, search and decision have the "same difficulty" — Karp reduced to each other.

Decision to Search – SAT

Let D be a decider for SAT. The following algorithm is a searcher for SAT.
Given a formula ϕ and a Boolean variable x in ϕ, let $\phi_{x=b}$ be the formula implied by fixing x to b.

Algorithm 15 (Searcher)

Input: a ϕ of variables $x_1 \ldots, x_\ell$.

1. If $D(\phi) = 0$, return $\perp$. If $\phi \equiv 1$, return $\emptyset$.
2. If $D(\phi_{x_1=1}) = 1$, return $(1, \text{Searcher}(\phi_{x_1=1}))$.
   Otherwise, return $(0, \text{Searcher}(\phi_{x_1=0}))$.

Decision to Search – CLIQUE

Let D be a decider for CLIQUE. The following algorithm is a searcher for CLIQUE.

Given a graph $G = (V, E)$:

- For $V' \subseteq v$, let $G_{-V'}$ be the graph implied by removing V' from G.
- For $v \in V$, let $N(v)$ be the neighbours of v in G (including v).

Algorithm 16 (Searcher)

Input: an integer k and graph G with vertices $V = \{v_1, \ldots, v_\ell\}$.

1. If $D(G, k) = 0$, return $\perp$. If $k = 0$, return $\emptyset$.
2. If $D(G_{-v_1}, k) = 1$, return $\text{Searcher}(G_{-\{v_1\}}, k)$.
   Otherwise, return $\{v_1\} \cup \text{Searcher}(G_{-(V \setminus N(v_1))}, k - 1)$.

Remark 17

Such reductions (known as downward self reductions), exist for all NP-Complete languages (why?), but seemingly not to all of NP.

Search Vs. Optimization Problems

- Search Problem: Given input x, find some y satisfying $R(x, y)$, or declare that none exist.
- Optimization Problem: Given input x, find some y with $(x, y) \in T$, that is the largest among all solutions (or smallest), or declare that none exist.
- Example 1: Given a graph G, find a vertex cover of smallest size possible.
- Example 2: Given a graph G, find a clique of largest size possible.

Coping with NP-Hardness

- Approximation algorithms for hard optimization problems.
- Randomized (coin flipping) algorithms.
- Fixed parameter algorithms.
- Heuristics.

These stand in the forefront of current algorithmic research, and could easily fill up three or four advanced courses.

Section 2

Coping with NP-Harness – Approximation

Approximation Algorithms

Solve efficiently, while giving a guarantee on the solution quality.

Approximation ratio:

- For minimization problem $\frac{cost(approx)}{cost(opt)}$
- For maximization problem $\frac{cost(opt)}{cost(approx)}$

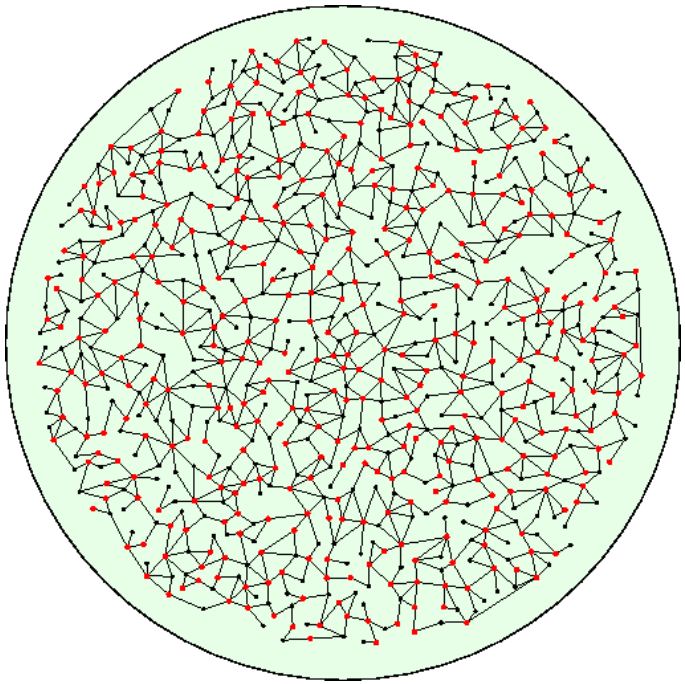
Fully Polynomial Approximation: $(1+\epsilon)$-approximation, where running time depends on ϵ.

Vertex Cover

Given a graph $G = (V, E)$ find the smallest vertex cover $C \subseteq V$ that contains at least one endpoint for every $e \in E$.

$$\text{VertexCover} = \{\langle G, k\rangle : \exists \text{ a vertex cover for } G \text{ of size } k\}$$

Recall that VertexCover $\in \mathcal{NPC}$ by a reduction from Independent Set (proved in recitation).



(figure from http://wwwbrauer.in.tum.de/gruppen/theorie/hard/vc1.png)

Approximating Algorithm for Vertex Cover (Gavril '74)

Algorithm 18 (Appx)

Input: A graph $G = (V, E)$

1. Set $C = \emptyset$
2. While there are edges in E:
   1. Choose any edge $(u, v) \in E$
   2. Add u and v to C
   3. Remove u and v from G.

Claim 19

Appx is a 2-approximation algorithm for vertex cover – C is at most twice as large as a minimum vertex cover.

Appx is a 2-Approximation for Vertex Cover

Proof:

- The cover C constructed from $|C|/2$ edges E_C of G
- No two edges of E_C share a vertex
- Any vertex cover (including the optimum) contain) at least one node from each $e \in E_C$ (otherwise an edge would not be covered).

  $\Longrightarrow$ $OPT(G) \geq |C|/2$ (or $\frac{|C|}{OPT} \leq 2$)

♣

Remark 20

Under some plausible complexity assumption, this factor 2-approximation cannot be improved.

Set Cover

Given a universe (i.e., set) U and a collection of m sets $S_i \subseteq U$, are there k indexes $\mathcal{I} \subseteq [k]$ that cover U (i.e., $\cup_{j \in \mathcal{I}} S_j = U$)?
Define

$$\text{SET-COVER} = \{\langle U, S_1, \ldots, S_m, k\rangle : \exists \mathcal{I} \subseteq [k] \text{ s.t. } \bigcup_{j \in \mathcal{I}} S_j = U\}$$

Note that

- SET-COVER $\in \mathcal{NP}$
- VertexCover $\leq_P$ SET-COVER

  $\Longrightarrow$ SET-COVER $\in \mathcal{NPC}$

Approximating Algorithm for Set Cover

Algorithm 21 (Appx)

Input: Sets $U, S_1, \ldots, S_m$.

1. Set $V_0 = U$ and $t = 1$.
2. While $V_t \neq \emptyset$
   1. Let $i_t = \arg\max_i |S_i \cap V_t|$
   2. $V_{t+1} = V_t \setminus S_{i_t}$
   3. $t = t + 1$
3. Output $C = \{i_1, \ldots, i_{t-1}\}$.

Claim 22

Appx is a $(\log |U|)$-approximation for set cover – $\mathcal{I}$ is at most $\log |U|$ as large as a minimum set cover.

Appx is a $(\log |U|)$-Approximation for Set Cover

Proof:

- Assume there are k subsets that cover U, and let $\mathcal{I}^*$ be their index set.
- In the t'th iteration of Appx, one of the subsets in $\mathcal{I}^*$ covers at least $|V_t|/k$ items in V_t.
- Hence, $|V_t| \leq (1 - \frac{1}{k}) \cdot |V_{t-1}| \leq (1 - \frac{1}{k})^{t-1} \cdot |U|$.
- For $t > k \cdot \log |U|$ it holds that $|V_t| < 1$, and hence Appx terminates.

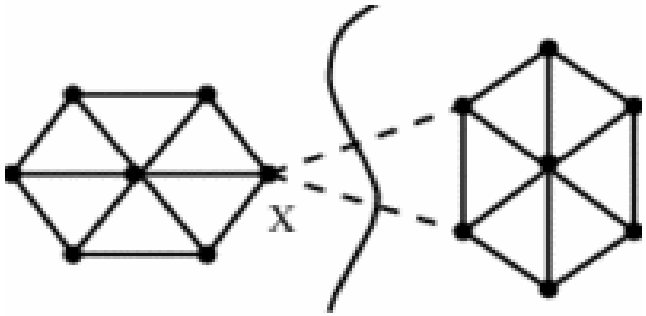
  $\Longrightarrow$ Appx approximation ratio $\log |U|$.

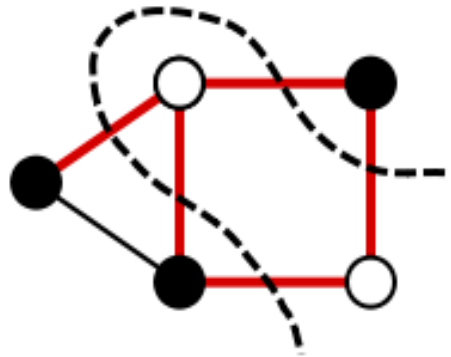
♣

Cuts in Graphs

Definition 23

$G = (V, E)$ be an undirected graph. For any partition of the nodes of into two sets, S and $V - S$, the set of edges between S and $V \setminus S$ is called a cut .





left pic from http://www.cs.sunysb.edu/∼algorith/files/edge-vertex-connectivity.shtml, right from wikipedia

Cuts Optimization

For cuts, both optimization problems make sense (in different contexts):

1. MIN-CUT: Find a partition that minimizes the number of edges between S and $V \setminus S$.
2. MAX-CUT: Find a partition that maximizes the number of edges between S and $V \setminus S$.

The two optimization problems have very different complexities:

1. MIN-CUT is tightly related to network flow, and has polynomial time algorithms.
2. MAX-CUT is NP-hard.

Approximation Algorithm for MAX-CUT

Consider the following local improvement strategy

Algorithm 24 (Appx)

1. Pick any partition S and $V \setminus S$
2. If the cut can be improved by moving any vertex from $V \setminus S$ to S, or vice-versa, do so.
3. Quit when no improvement is possible (local maximum reached).

Running time:

- Any cut has at most $|E|$ edges
  - $\Longrightarrow$ at most $|E|$ improvements are possible
  - $\Longrightarrow$ Appx is polynomial time.

Claim 25

Appx is a 2-approximation algorithm for MAX-CUT.

Cut Edges and Non-Cut Edges

We use the following definitions with respect to a fixed cut C,

- The cut partitions the edges into cut edges E_C, and non-cut edges E_N.
- Let c_v be the number of cut edges from node v.
- Let n_v be the number of non-cut edges from v.

Appx is a 2-Approximation for MAX-CUT

Proof: When Appx terminates:

- $c_v \geq n_v$ for every node v (why?)
- $\implies \sum_{v \in V} c_v \geq \sum_{v \in V} n_v$
- $\sum_{v \in V} c_v = 2|E_C|$ and $\sum_{v \in V} n_v = 2|E_N|$ (each edge is counted twice).
- $\implies |E_C| \geq |E_N|$.
- $\implies 2|E_C| \geq |E_N| + |E_C| = |E|$
- $\implies |E_C| \geq |E|/2$.
- Since $|E| \geq OPT$, it holds that $|E_C| \geq OPT/2$

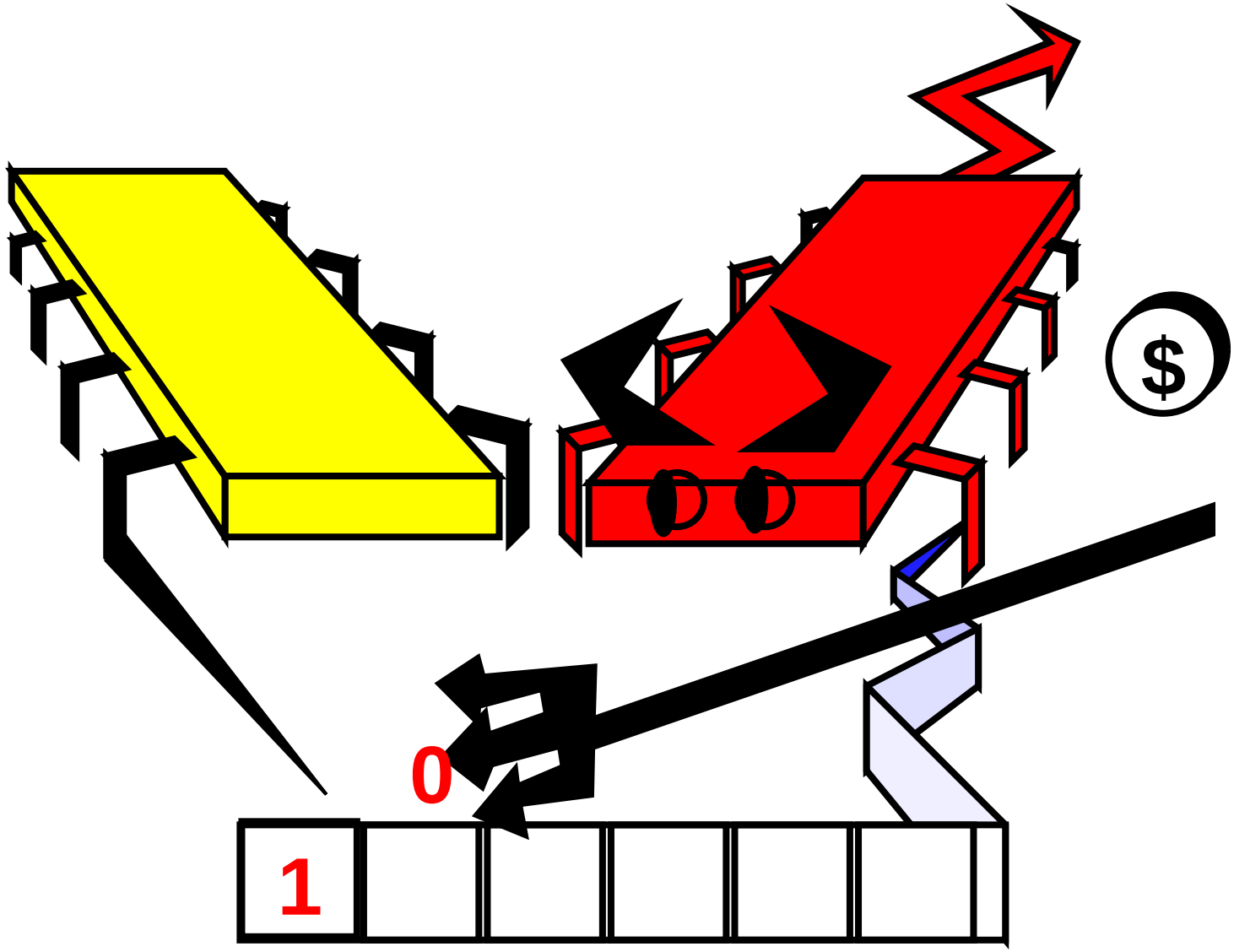
  Hence, Appx is 2-approximation (i.e., $\frac{OPT}{|E_C|} \leq 2$)

♣

Section 3

Coping with NP-Hardness – Randomization

Randomized TMs



Randomized Computation

We can sometimes use randomization to solve problems that are difficult to solve deterministically.

Examples:

- MAX-CUT
- MAX-3SAT

Randomized Algorithm for MAX-CUT

- We give a randomized 2-approximation for MAX-CUT[2]
- **Strategy**: Each node v with probability $\frac{1}{2}$ is in S.
- **Analysis**: For each edge (u, v) with probability $\frac{1}{2}$ is in the cut.
- Expected size of cut is $|E|/2$.
- Clearly, in OPT we cut at most $|E|$ edges.

[2] On expectation.

Randomized Algorithm for MAX-3SAT

- We give a randomized $8/7$-approximation for MAX-3SAT[3]
- **Strategy**: Each variable x_i is with probability $\frac{1}{2}$ set $x_i = 1$ (and otherwise set $x_i = 0$).
- **Analysis**: Each clause C is satisfied with probability $\frac{7}{8}$.
- Expected number of satisfied clauses is at least $\frac{7}{8}m$, where m is the number of clauses.
- Clearly, in OPT we satisfy at most m clauses.

[3] On expectation.

Section 4

Coping with NP-Hardness – Fixed Parameter

Fixed Parameter

- Main idea: Many NP-complete problems have a natural parameter r.
- Examples: clique size (CLIQUE), number of variables (SAT), size of cover (VertexCover), magnitude of integers (SUBSET-SUM).
- The hardness depends on the parameter r.
- Get an algorithm which runs in $O(f(r) \cdot \text{poly}(n))$, “caring not" about f.

- Simple example SAT:
  - SAT has a natural parameter k – the number of variables
  - Easy to show $O(2^k n)$-time solution , where n is the formula size.

Fixed Parameter Algorithm for Vertex Cover

The parameter: k — the size of the optimal cover.
Fixed parameter algorithm runs in time $O(2^k \cdot |E|)$.

Algorithm 26 (D)

Input: Graph $G = (V, E)$ and integer k:

1. If $E = \emptyset$, return TRUE. If $k = 0$, return FALSE.
2. Pick edge $(u, v) \in E$.
   If $D(G_{-u}, k-1) \vee D(G_{-v}, k-1)$, return TRUE.
   Otherwise, return FALSE.

Time analysis:

- The recursion depth is at most k.
- At least one of the recursive paths is successful.

Section 5

Coping with NP-Hardness – Heuristics

Heuristics

http://image.examiner.com/images/blog/wysiwyg/image/beast.jpg

Heuristics

Definition:

Main Entry heuristic
Pronunciation: hyu-'ris-tik
Function: adjective
Etymology: German heuristisch, from New Latin heuristicus, from Greek heuriskein – to discover;

Involving or serving as an aid to learning, discovery, or problem-solving by experimental and especially trial-and-error methods (heuristic techniques; a heuristic assumption);
also: of or relating to exploratory problem-solving techniques that utilize self-educating techniques (as the evaluation of feedback) to improve performance (a heuristic computer program)

Heuristics

Heuristics are widely used in almost every area with hard optimization problems. They are typically based on solid intuition, but their run-time analysis "in practice" is beyond current knowledge.

Examples: Simulated annealing, genetic (evolutionary) algorithms, neural networks.

When all else fails, a smart heuristic may do wonders.

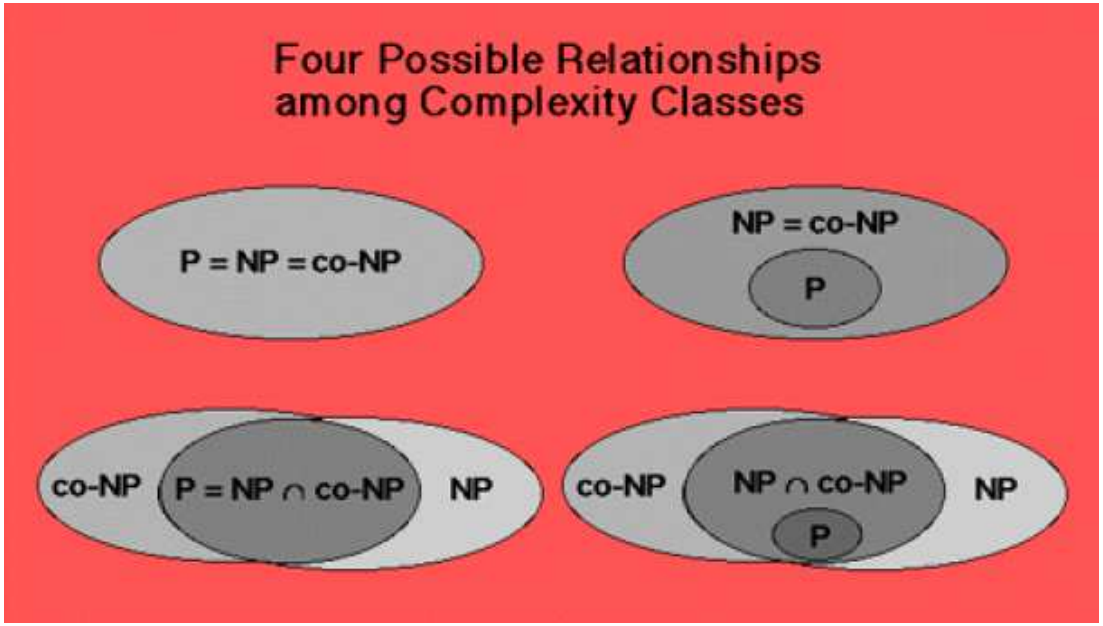
Part III

Concluding Remarks

Ladies and Gentlemen, Boys and Girls

We have just ended the third part of the course:

Introduction to Computational Complexity (no more).



And with it, naturally, we've ended the whole course. We hope you have enjoyed your flight, and look forward to meeting you in the future (but not in Moed B :-).

The Dreaded Exam

- All material covered in class and recitations, from three parts of course.
- Both multiple choice ("closed") and "open" questions.
- You can bring and use two double sided A4 (normal size) pages marked with your name
- Piece of cake.