Computational Models - Lecture 1[1]

Handout Mode

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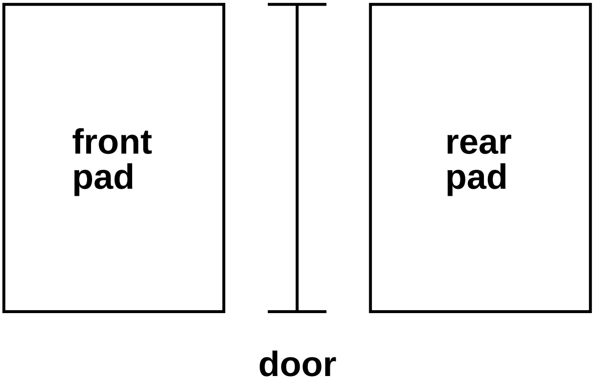
[1]Based on frames by Benny Chor, Tel Aviv University, modifying frames by Maurice Herlihy, Brown University.

Finite Automata

Topics covered:

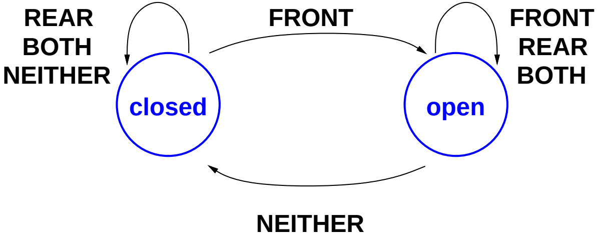
- formal definition of finite automata
- deterministic vs. non-deterministic finite automata
- regular languages
- operations on regular languages
- regular expressions
- pumping lemma

Example: A One-Way Automatic Door



- open when person approaches
- hold open until person clears
- don't open when someone standing behind door

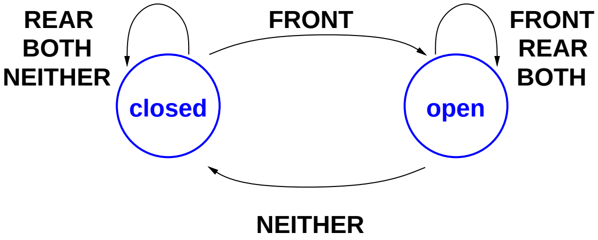
The Automatic Door as DFA



- States:
  - OPEN
  - CLOSED
- Sensor:
  - FRONT: someone on front pad
  - REAR: someone on rear pad
  - BOTH: someone(s) on both pads
  - NEITHER no one on either pad.

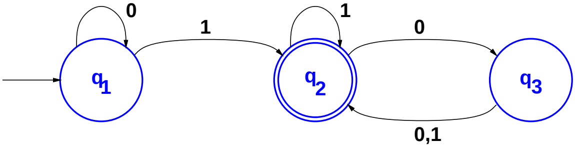
The Automatic Door as DFA

DFA is Deterministic Finite Automata



	neither	front	rear	both
closed	closed	open	closed	closed
open	closed	open	open	open

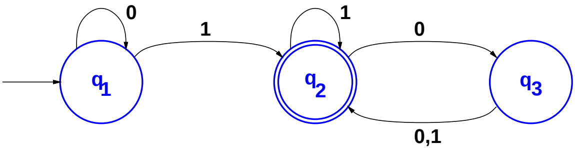
DFA: Informal Definition



The machine M_1:

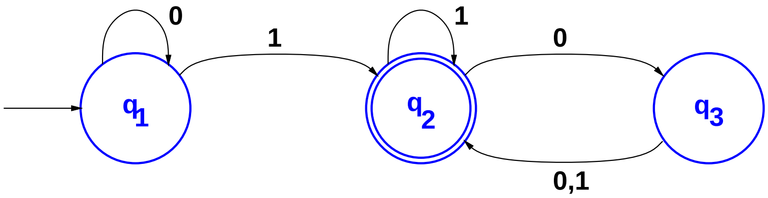
- states: q_1, q_2, and q_3.
- start state: q_1 (arrow from “outside”).
- accept state: q_2 (double circle).
- state transitions: arrows tagged with letters.

DFA: Informal Definition (cont.)



- On an input string
  - DFA begins in start state q_1
  - after reading each symbol, DFA makes state transition with matching label.
- After reading last symbol, DFA produces output:
  - accept if DFA is an accepting state.
  - reject otherwise.

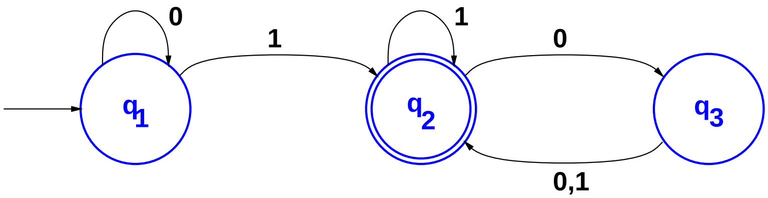
Informal Definition - Example



What happens on input strings

- 1101
- 0010
- 01100
- In general?!

Informal Definition



This DFA accepts

- all input strings that end with a 1
- all input strings that contain at least one 1, and end with an even number of 0's
- no other strings

Languages, words and Alphabets

Definition 1

An alphabet Σ is a finite set of letters.

- $\Sigma = \{a, b, c, \ldots, z\}$ – the English alphabet.
- $\Sigma = \{\alpha, \beta, \gamma, \ldots, \zeta\}$ – the Greek alphabet.
- $\Sigma = \{0, 1\}$ – the binary alphabet.
- $\Sigma = \{0, 1, \ldots, 9\}$ – the digital alphabet.

Definition 2

A word (string) is a finite sequence of letters.

The collection of all strings over Σ is denoted by Σ^*.

For the binary alphabet, ε, 1, 0, 000000000, 1111111000 are all members of Σ^*.

Languages and Examples

Definition 3

A language over Σ is a subset $\mathcal{L} \subseteq \Sigma^*$.

For example

- Modern English.
- Ancient Greek.
- All prime numbers, written using digits.
- $A = \{w : w \text{ has at most seventeen 0's}\}$.
- $B = \{0^n 1^n : n \geq 0\}$.
- $C = \{w : w \text{ has an equal number of 0's and 1's}\}$.

Languages and DFA

Definition 4

$\mathcal{L}(M)$, the language of a DFA M, is the set of strings $\mathcal{L}$ that M accepts, $\mathcal{L}(M) = \mathcal{L}$.

Note that

- M may accept many strings, but
- M accepts only one language.

What language does M accept if it accepts no strings?

Definition 5

A language is called regular if some deterministic finite automaton accepts it.

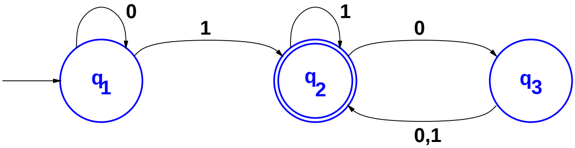
Formal Definitions

Definition 6

A deterministic finite automaton (DFA) is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

- Q is a finite set called the states,
- Σ is a finite set called the alphabet,
- $\delta : Q \times \Sigma \to Q$ is the transition function,
- $q_0 \in Q$ is the start state, and
- $F \subseteq Q$ is the set of accept states.

Back to M_1



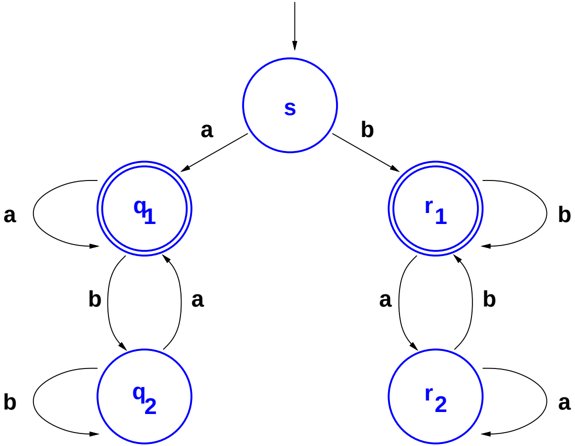
$M_1 = (Q, \Sigma, \delta, q_1, F)$ where

- $Q = \{q_1, q_2, q_3\}$, $\Sigma = \{0, 1\}$,
- the transition function δ is

	0	1
q_1	q_1	q_2
q_2	q_3	q_2
q_3	q_2	q_2

- q_1 is the start state, and $F = \{q_2\}$.

Another Example



A Formal Model of Computation

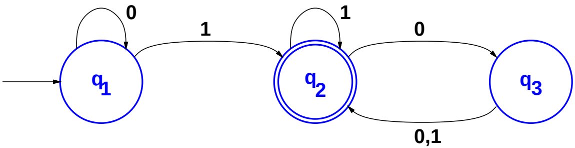
- Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA, and
- let $w = w_1 w_2 \cdots w_n$ be a string over Σ.

We say that M accepts w if there is a sequence of states $r_0, \ldots, r_n$ ($r_i \in Q$) such that

- $r_0 = q_0$
- $\delta(r_i, w_{i+1}) = r_{i+1}$, $0 \leq i < n$
- $r_n \in F$

Extend δ to work on strings: for $\delta(q, y\sigma) = \delta(\delta(q, y), \sigma)$, for $|y| \geq 1$.

Proving a language of a DFA



Theorem 7

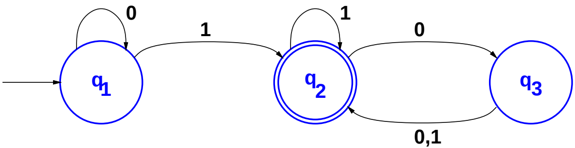
$\mathcal{L}(M_1) = \{w10^{2k} : k \geq 0, w \in \Sigma^*\}$

Proof: For each state $q_i \in Q$ let $\mathcal{L}(q_i) = \{x : \delta(q_1, x) = q_i\}$.
Induction hypothesis:

- $\mathcal{L}(q_1) = \{0^k : k \geq 0\}$.
- $\mathcal{L}(q_2) = \{w10^{2k} : k \geq 0, w \in \Sigma^*\}$.
- $\mathcal{L}(q_3) = \{w10^{2k+1} : k \geq 0, w \in \Sigma^*\}$.

Induction Basis: $x = \epsilon$, $|x| = 0$.
Induction Step: $x = y\sigma$, $|x| = j + 1$, $|y| = j$ and $\sigma \in \Sigma$.
Prove: $x \in \mathcal{L}(q_i)$ iff $\delta(q_1, x) = q_i$.

Proving a language of a DFA



Prove: $x \in \mathcal{L}(q_i)$ iff $\delta(q_1, x) = q_i$.

Claim: If $x \in \mathcal{L}(q_1)$ Then $\delta(q_1, x) = q_1$

If $x \in \mathcal{L}(q_1)$ Then $x = 0^{j+1}$, $y = 0^j$ and $\sigma = 0$.

$y \in \mathcal{L}(q_1)$, therefore, by the induction hypothesis, $\delta(q_1, y) = q_1$.

By definition, $\delta(q_1, 0) = q_1$.

Therefore $\delta(q_1, x) = \delta(\delta(q, y), \sigma) = \delta(q_y, \sigma) = \delta(q_1, 0) = q_1$.

Claim: If $\delta(q_1, x) = q_1$ Then $x \in \mathcal{L}(q_1)$

Let $q_y = \delta(q_1, y)$.

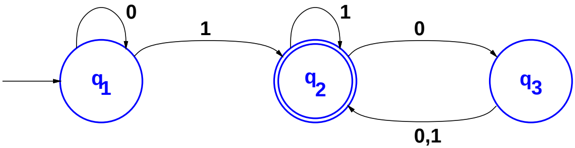
$\delta(q_1, x) = \delta(q_y, \sigma) = q_1$.

This can happen only if $q_y = q_1$ and $\sigma = 0$.

By the induction hypothesis, if $q_y = q_1$ then $y = 0^j$.

Therefore $x = y\sigma = 0^{j+1} \in \mathcal{L}(q_1)$

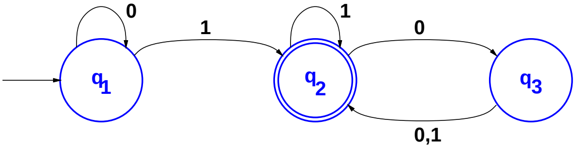
Proving a language of a DFA



Claim: If $x \in \mathcal{L}(q_2)$ Then $\delta(q_1, x) = q_2$
If $x \in \mathcal{L}(q_2)$ Then $x = w10^{2k}$, $k \geq 1$, or $x = w1$.
If $x = w1$ then $y = w$ and $\sigma = 1$.
Since from any q_i, $\delta(q_i, 1) = q_2$ then $\delta(q_1, x) = q_2$.
If $x = w10^{2k}$ then $y = w10^{2k-1}$ and $\sigma = 0$. Then $y \in \mathcal{L}(q_3)$.
By the induction hypothesis $\delta(q_1, y) = q_3$.
Then $\delta(q_3, 0) = q_2$.

Claim: If $\delta(q_1, x) = q_2$ Then $x \in \mathcal{L}(q_2)$
Let $q_y = \delta(q_1, y)$ and $\delta(q_1, x) = \delta(q_y, \sigma) = q_2$.
This can happen if either $\sigma = 1$ or $q_y = q_3$ and $\sigma = 0$.
If $\sigma = 1$ then $x \in \mathcal{L}(q_2)$.
If $\sigma = 0$ and $q_y = q_3$, then, by the induction hypothesis, $y = w10^{2k+1}$.
Therefore $x = y\sigma = w10^{2k+1}0 \in \mathcal{L}(q_2)$

Proving a language of a DFA



Claim: If $x \in \mathcal{L}(q_3)$ Then $\delta(q_1, x) = q_3$

If $x \in \mathcal{L}(q_3)$ Then $x = w10^{2k+1}$, and $y = w10^{2k}$ and $\sigma = 0$.

Then $y \in \mathcal{L}(q_2)$, and by the induction hypothesis $\delta(q_1, y) = q_2$.

Therefore, $\delta(q_2, 0) = q_3$.

Claim: If $\delta(q_1, x) = q_3$ Then $x \in \mathcal{L}(q_3)$

Let $q_y = \delta(q_1, y)$ and $\delta(q_1, x) = \delta(q_y, \sigma) = q_3$.

This can happen only if $\sigma = 0$ or $q_y = q_2$.

By the induction hypothesis, $y = w10^{2k}$.

Therefore $x = y\sigma = w10^{2k}0 \in \mathcal{L}(q_3)$

♣

Examples of regular languages

Assume $\Sigma = \{0, 1\}$.

- Odd number of 1:, i.e., $\{w | \#_1(w) \pmod 2 = 1\}$.
- Sequence of 0 followed by sequence of 1, i.e., $\{0^m 1^n | m, n \geq 0\}$.
- Any finite language.

The Regular Operations

Let A and B be languages.

The union operation:

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

The concatenation operation:

$$A \| B = \{xy : x \in A \text{ and } y \in B\}$$

The star operation:

$$A^* = \{x_1 x_2 \ldots x_k : k \geq 0 \text{ and each } x_i \in A\}$$

The Regular Operations – Examples

Let $A = \{\text{good}, \text{bad}\}$ and $B = \{\text{boy}, \text{girl}\}$.

Union

$$A \cup B = \{\text{good}, \text{bad}, \text{boy}, \text{girl}\}$$

Concatenation

$$A \| B = \{\text{goodboy}, \text{goodgirl}, \text{badboy}, \text{badgirl}\}$$

Star

$$A^* = \{\varepsilon, \text{good}, \text{bad}, \text{goodgood}, \text{goodbad}, \text{badbad}, \text{badgood}, \ldots\}$$

Claim: Closure Under Union

If A_1 and A_2 are regular languages, so is $A_1 \cup A_2$.

Approach to Proof:

- some M_1 accepts A_1
- some M_2 accepts A_2
- construct M that accepts $A_1 \cup A_2$.

Attempted Proof Idea:

- first simulate M_1, and
- if M_1 doesn't accept, then simulate M_2.

What's wrong with this?
Fix: Simulate both machines simultaneously.

Closure Under Union: Correct Proof

- Suppose $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ accepts $\mathcal{L}_1$,
- and $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ accepts $\mathcal{L}_2$.

Define M as follows (M will accept $\mathcal{L}_1 \cup \mathcal{L}_2$):

- $Q = Q_1 \times Q_2$.
- Σ is the same.
- For each $(r_1, r_2) \in Q$ and $a \in \Sigma$,
  $\delta((r_1, r_2), a) = (\delta_1(r_1, a), \delta_2(r_2, a))$
- $q_0 = (q_1, q_2)$
- $F = \{(r_1, r_2) \colon r_1 \in F_1$ or $r_2 \in F_2\}$.
- Formal proof (next slide) ♣

(hey, why not choose $F = F_1 \times F_2$?)

Correctness of the construction

Theorem 8

$\mathcal{L}(M) = \mathcal{L}(M_1) \cup \mathcal{L}(M_2)$.

Proof:
Induction hypothesis: $\delta((q_1, q_2), x) = (\delta_1(q_1, x), \delta_2(q_2, x))$.
Induction Basis: $x = \epsilon$, $|x| = 0$. Trivial
Induction Step: $x = y\sigma$, $|x| = i + 1$, $|y| = i$ and $\sigma \in \Sigma$.
Follows from the definition of δ.

Completing the proof:
If $x \in \mathcal{L}(M_1)$ then $\delta_1(q_1, x) = r_1 \in F_1$.
$\delta((q_1, q_2), x) = (r_1, r_2) \in F$.
(Similar if $x \in \mathcal{L}(M_2)$.)
If $x \in \mathcal{L}(M)$ then $\delta((q_1, q_2), x) = (r_1, r_2) \in F$.
Then, $\delta_i(q_i, x) = r_i$, $i \in \{1, 2\}$.
Since $(r_1, r_2) \in F$ either $r_1 \in F_1$ or $r_2 \in F_2$.

What About Concatenation?

Theorem 9

If $\mathcal{L}_1$, $\mathcal{L}_2$ are regular languages, so is $\mathcal{L}_1 \| \mathcal{L}_2$.

Example: $\mathcal{L}_1 = \{\text{good}, \text{bad}\}$ and $\mathcal{L}_2 = \{\text{boy}, \text{girl}\}$.

$$\mathcal{L}_1 \| \mathcal{L}_2 = \{\text{goodboy}, \text{goodgirl}, \text{badboy}, \text{badgirl}\}$$

This is much harder to prove.

Idea: Simulate M_1 for a while, then switch to M_2.

Problem: But when do you switch? This leads us into non-determinism.